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The Inhomogeneous Quantum Invariance Group of The Two Parameter Deformed Boson Algebra

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Abstract We consider two parameter deformed boson algebra and investigate the inhomogeneous invariance quantum group of this system. We find the R-matrix which collects all information about the non-commuting structure of the quantum group. We extend our study to the d-dimensional case.

Keywords Quantum groups · Two parameter deformed boson algebra

1 Introduction

Group theory which can be interpreted as the mathematical theory of symmetry has a crucial importance in physics. Although Lie groups and Lie algebras are very helpful in order to understand many concepts in physics, they are not sufficient to study the theory of quantum integrable systems.

In 1980's, it was realized that quantum inverse scattering method which is used to study the quantum integrable systems needs a further generalization for classical Lie groups and Lie algebras [1]. First achievement for this generalization was performed by obtaining the q-analog of $SU(2)$ [2, 3]. In 1990's, Macfarlane and Biedenharn showed that one can find $SU_q(2)$ by considering q-deformed boson algebras [4, 5]. This situation has rekindled the interest in the deformed oscillator systems [6–12].

In this study, we consider a boson algebra including two deformation parameters. To make calculations simple, we first investigate the inhomogeneous invariance quantum group

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for the following algebraic structure

$$aa^* - q^2 a^* a = H, \tag{1}$$

$$aH = r^2 Ha, \tag{2}$$

where H is a Hermitean operator. Then we construct the R matrix for this system.

In Sect. 3, we extend our study from one dimensional case to the two dimensional case by considering the following two parameter deformed algebra [13]

$$a_1 a_1^* - q^2 a_1^* a_1 = H, \tag{3}$$

$$a_2 a_2^* - q^2 a_2^* a_2 = H, \tag{4}$$

$$a_1 H = r^2 H a_1, \tag{5}$$

$$a_2 H = r^2 H a_2, \tag{6}$$

$$a_1 a_2 = a_2 a_1, \tag{7}$$

$$a_1 a_2^* = r^2 a_2^* a_1. \tag{8}$$

We obtain that the homogeneous part of the inhomogeneous invariance quantum group of this system is given by the multiparameter deformed general linear group of Schirrmacher [14] where deformation parameters take some special values. Finally, we do same calculations for the d-dimensional two parameter deformed boson algebra.

2 Quantum Invariance Group for One Dimensional Two Parameter Deformed Boson Algebra

In order to find the inhomogeneous quantum invariance group of the (q,r)-deformed boson algebra

$$aa^* - q^2 a^* a = H, \tag{9}$$

$$aH = r^2 Ha, \tag{10}$$

it is possible to consider the following 4×4 square quantum matrix M

$$M = \begin{pmatrix} \alpha & \beta & \eta & \gamma \\ \beta^* & \alpha^* & \eta^* & \gamma^* \\ 0 & 0 & \chi_3 & \chi_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

as a transformation matrix [15–21] such that it acts on 4×1 column vector V by matrix co-multiplication. The elements of the column vector V are the generators of the algebra defined by (9) and (10), namely,

$$V = \begin{pmatrix} a \\ a^* \\ H \\ 1 \end{pmatrix}. \tag{12}$$

Thus, the transformed generators can be written as

$$a' = \alpha \otimes a + \beta \otimes a^* + \eta \otimes H + \gamma \otimes 1, \tag{13}$$

$$a^{*\prime} = \beta^* \otimes a + \alpha^* \otimes a^* + \eta^* \otimes H + \gamma^* \otimes 1, \tag{14}$$

$$H' = \chi_3 \otimes H + \chi_4 \otimes 1. \tag{15}$$

The invariance of the system under the above transformations requires the special commutation relations among the elements of the matrix M. That is to say, the elements of matrix M belong to an associative algebra which satisfies the following commutation relations

$$\alpha\beta = q^{-2}\beta\alpha, \tag{16}$$

$$\alpha\eta = r^{-2}\eta\alpha, \tag{17}$$

$$\alpha\gamma = \gamma\alpha, \tag{18}$$

$$\alpha\chi_3 = \chi_3\alpha, \tag{19}$$

$$\alpha\chi_4 = r^2\chi_4\alpha, \tag{20}$$

$$\beta\eta = r^2\eta\beta, \tag{21}$$

$$\beta\gamma = \gamma\beta, \tag{22}$$

$$\beta\chi_3 = r^4\chi_3\beta, \tag{23}$$

$$\beta\chi_4 = r^2\chi_4\beta, \tag{24}$$

$$\eta\gamma = \gamma\eta, \tag{25}$$

$$\eta\chi_3 = r^2\chi_3\eta, \tag{26}$$

$$\eta\chi_4 = r^2\chi_4\eta, \tag{27}$$

$$\gamma\chi_3 = r^2\chi_3\gamma, \tag{28}$$

$$\gamma\chi_4 = r^2\chi_4\gamma, \tag{29}$$

$$\chi_3\chi_4 = \chi_4\chi_3, \tag{30}$$

$$\alpha\beta^* = q^2\beta^*\alpha, \tag{31}$$

$$\alpha\alpha^* = \alpha^*\alpha, \tag{32}$$

$$\alpha\eta^* = \frac{q^2}{r^2}\eta^*\alpha, \tag{33}$$

$$\alpha\gamma^* = q^2\gamma^*\alpha, \tag{34}$$

$$\beta\beta^* = q^4\beta^*\beta, \tag{35}$$

$$\beta\eta^* = q^2r^2\eta^*\beta, \tag{36}$$

$$\beta\gamma^* = q^2\gamma^*\beta, \tag{37}$$

$$\eta\eta^* = q^2\eta^*\eta, \tag{38}$$

$$\eta\gamma^* - q^2\gamma^*\eta = \frac{1}{2}(\chi_3 + q^2\beta^*\beta - \alpha\alpha^*), \tag{39}$$

$$\gamma\gamma^* - q^2\gamma^*\gamma = \chi_4. \tag{40}$$

These relations can be obtained not only from the invariance of (9) and (10) under the transformations (13) to (15) but also by imposing that the algebraic structure generated by the elements of the matrix M should be a Hopf algebra [22, 23]. To examine whether this algebraic structure is a Hopf algebra or not, one can define the coproduct with a matrix multiplication

$$\Delta(M) = M\dot{\otimes}M \tag{41}$$

which makes possible to write the coproduct of α, β, η, γ, χ_3 and χ_4 as

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \beta^*, \tag{42}$$

$$\Delta(\beta) = \alpha \otimes \beta + \beta \otimes \alpha^*, \tag{43}$$

$$\Delta(\eta) = \alpha \otimes \eta + \beta \otimes \eta^* + \eta \otimes \chi_3, \tag{44}$$

$$\Delta(\gamma) = \alpha \otimes \gamma + \beta \otimes \gamma^* + \eta \otimes \chi_4 + \gamma \otimes 1, \tag{45}$$

$$\Delta(\chi_3) = \chi_3 \otimes \chi_3, \tag{46}$$

$$\Delta(\chi_4) = \chi_3 \otimes \chi_4 + \chi_4 \otimes 1. \tag{47}$$

The invariance of the commutation relations satisfied by the coproducts is not sufficient to say that this algebraic structure is a Hopf algebra. To say it, one should also find the inverse of matrix M where the counit and antipode are defined as

$$\epsilon(M) = 1, \tag{48}$$

$$S(M) = M^{-1}. \tag{49}$$

Transformation matrix M can be written in a block form

$$\begin{pmatrix} \alpha & \beta & \eta & \gamma \\ \beta^* & \alpha^* & \eta^* & \gamma^* \\ 0 & 0 & \chi_3 & \chi_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & \Gamma \\ 0 & B \end{pmatrix}, \tag{50}$$

where A is the 2×2 square matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \tag{51}$$

which forms the homogeneous part of the transformation matrix M. Γ and B are also the 2×2 square matrices, namely,

$$\Gamma = \begin{pmatrix} \eta & \gamma \\ \eta^* & \gamma^* \end{pmatrix}, \tag{52}$$

$$B = \begin{pmatrix} \chi_3 & \chi_4 \\ 0 & 1 \end{pmatrix}. \tag{53}$$

This makes it possible to write

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\Gamma B^{-1} \\ 0 & B^{-1} \end{pmatrix}, \tag{54}$$

since $MM^{-1} = M^{-1}M = 1$ is satisfied. The elements of matrix B commute among themselves. Inverse of B is an ordinary matrix inverse. Therefore, it is appropriate to focus on the matrix A in order to investigate the inverse of matrix M. The elements of matrix A satisfy the following commutation relations

$$\alpha\beta = q^{-2}\beta\alpha, \tag{55}$$

$$\alpha\beta^* = q^2\beta^*\alpha, \tag{56}$$

$$\alpha\alpha^* = \alpha^*\alpha, \tag{57}$$

$$\beta\beta^* = q^4\beta^*\beta, \tag{58}$$

which implies that

$$A^{-1} = \begin{pmatrix} \alpha^* & -q^2\beta \\ -q^{-2}\beta^* & \alpha \end{pmatrix} D^{-1}, \tag{59}$$

where

$$D = \alpha\alpha^* - q^{-2}\beta\beta^*. \tag{60}$$

The commutation relations from (55) to (58) are the same with the commutation relations written for $GL_{p,q}(2)$ with $p \to q^{-2}$ and $q \to q^2$ [24]. Therefore, one can say that the homogeneous part of the transformation matrix M is an element of $GL_{q^{-2},q^2}(2)$.

Since quantum groups are usually considered to be quasi-triangular Hopf Algebras, the algebraic relations satisfied by the elements of the transformation matrix M satisfy [24, 25]

$$R(M \otimes 1)(1 \otimes M) = (1 \otimes M)(M \otimes 1)R. \tag{61}$$

We find that R is the $4^2 \times 4^2$ square matrix,

$$R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{q^{-2}}{2} & 0 & 0 & -\frac{q^{-2}}{2} & 0 \\
0 & 0 & r^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{62}$$

and stores up all information about the algebraic structure of the invariance group of the (q, r)-deformed boson algebra.

3 Quantum Invariance Group for Multi-dimensional Two Parameter Deformed Boson Algebra

For two bosons, (q,r)-deformed boson algebra $\mathcal{A}_2$ can be written as

$$a_1 a_1^* - q^2 a_1^* a_1 = H, \tag{63}$$

$$a_2 a_2^* - q^2 a_2^* a_2 = H, \tag{64}$$

$$a_1 H = r^2 H a_1, \tag{65}$$

$$a_2 H = r^2 H a_2, \tag{66}$$

$$a_1 a_2 = a_2 a_1, \tag{67}$$

$$a_1 a_2^* = r^2 a_2^* a_1. \tag{68}$$

The symmetry property of this system can be studied by considering following transformations

$$a_i' = \alpha_{ik} \otimes a_k + \beta_{ik} \otimes a_k^* + \eta_i \otimes H + \gamma_i \otimes 1, \tag{69}$$

$$a_i^{*\prime} = \beta_{ik}^* \otimes a_k + \alpha_{ik}^* \otimes a_k^* + \eta_i^* \otimes H + \gamma_i^* \otimes 1, \tag{70}$$

$$H' = \chi_3 \otimes H + \chi_4 \otimes 1. \tag{71}$$

Here, i takes the values 1 and 2 whereas the summation index k runs from 1 to 2. The invariance of (63) to (68) under the above transformations requires that α_{ij}, β_{ij}, η_i, γ_i, α_{ij}^*, β_{ij}^*, η_i^*, γ_i^*, χ_3 and χ_4 satisfy the following algebraic relations

$$\alpha_{ik}\alpha_{jm} = \alpha_{jm}\alpha_{ik}, \tag{72}$$

$$\alpha_{ik}\beta_{jm} = f_1(q,r)\beta_{jm}\alpha_{ik}, \tag{73}$$

$$\alpha_{ik}\eta_j = r^{-2}\eta_j\alpha_{ik}, \tag{74}$$

$$\alpha_{ik}\gamma_j = \gamma_j\alpha_{ik}, \tag{75}$$

$$\alpha_{ik}\chi_3 = \chi_3\alpha_{ik}, \tag{76}$$

$$\alpha_{ik}\chi_4 = r^2\chi_4\alpha_{ik}, \tag{77}$$

$$\beta_{ik}\beta_{jm} = \beta_{jm}\beta_{ik}, \tag{78}$$

$$\beta_{ik}\eta_j = r^2\eta_j\beta_{ik}, \tag{79}$$

$$\beta_{ik}\gamma_j = \gamma_j\beta_{ik}, \tag{80}$$

$$\beta_{ik}\chi_3 = r^4\chi_3\beta_{ik}, \tag{81}$$

$$\beta_{ik}\chi_4 = r^2\chi_4\beta_{ik}, \tag{82}$$

$$\eta_i\eta_j = \eta_j\eta_i, \tag{83}$$

$$\eta_i\gamma_j - \gamma_j\eta_i = \frac{1}{2}\sum_{k=1}^{2}(\alpha_{jk}\beta_{ik} - \alpha_{ik}\beta_{jk}), \tag{84}$$

$$\eta_i\chi_3 = r^2\chi_3\eta_i, \tag{85}$$

$$\eta_i \chi_4 = r^2 \chi_4 \eta_i, \tag{86}$$

$$\gamma_i \gamma_j = \gamma_j \gamma_i, \tag{87}$$

$$\gamma_i \chi_3 = r^2 \chi_3 \gamma_i, \tag{88}$$

$$\gamma_i \chi_4 = r^2 \chi_4 \gamma_i, \tag{89}$$

$$\alpha_{ik} \beta^*_{jm} = f_2(q,r) \beta^*_{jm} \alpha_{ik}, \tag{90}$$

$$\alpha_{ik} \alpha^*_{jm} = f_3(q,r) \alpha^*_{jm} \alpha_{ik}, \tag{91}$$

$$\alpha_{ik} \eta^*_j = r^{-2} f_2(q,r) \eta^*_j \alpha_{ik}, \tag{92}$$

$$\alpha_{ik} \gamma^*_j = f_2(q,r) \gamma^*_j \alpha_{ik}, \tag{93}$$

$$\beta_{ik} \beta^*_{jm} = f_4(q,r) \beta^*_{jm} \beta_{ik}, \tag{94}$$

$$\beta_{ik} \eta^*_j = r^2 f_2(q,r) \eta^*_j \beta_{ik}, \tag{95}$$

$$\beta_{ik} \gamma^*_j = f_2(q,r) \gamma^*_j \beta_{ik}, \tag{96}$$

$$\eta_i \eta^*_j = f_2(q,r) \eta^*_j \eta_i, \tag{97}$$

$$\eta_i \gamma^*_j - f_2(q,r) \gamma^*_j \eta_i = \chi_3 \delta_{ij} + f_2(q,r) \sum_{k=1}^{2} \beta^*_{jk} \beta_{ik} - \alpha_{ik} \alpha^*_{jk}, \tag{98}$$

$$\gamma_i \gamma^*_j - f_2(q,r) \gamma^*_j \gamma_i = \chi_4 \delta_{ij}, \tag{99}$$

where

$$f_1(q,r) = \begin{cases} q^{-2} & k = m \\ r^{-2} & k \neq m, \end{cases}$$

$$f_2(q,r) = \begin{cases} q^2 & i = j \\ r^2 & i \neq j, \end{cases}$$

$$f_3(q,r) = f_1(q,r) f_2(q,r) = \begin{cases} 1 & i = j \text{ and } k = m \\ q^2 r^{-2} & i = j \text{ and } k \neq m \\ q^{-2} r^2 & i \neq j \text{ and } k = m \\ 1 & i \neq j \text{ and } k \neq m, \end{cases}$$

$$f_4(q,r) = \frac{f_2(q,r)}{f_1(q,r)} = \begin{cases} q^4 & i = j \text{ and } k = m \\ q^2 r^2 & i = j \text{ and } k \neq m \\ q^2 r^2 & i \neq j \text{ and } k = m \\ r^4 & i \neq j \text{ and } k \neq m. \end{cases}$$

In order to study the Hopf algebra structure of the above algebraic system, one can define coproduct, counit and antipode as

$$\Delta(T) = T \dot{\otimes} T, \tag{100}$$

$$\epsilon(T) = 1, \tag{101}$$

$$S(T) = T^{-1}. \tag{102}$$

Here T is the transformation matrix which can be written

$$T = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} & \eta_1 & \gamma_1 \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} & \eta_2 & \gamma_2 \\ \beta^*_{11} & \beta^*_{12} & \alpha^*_{11} & \alpha^*_{12} & \eta^*_1 & \gamma^*_1 \\ \beta^*_{21} & \beta^*_{22} & \alpha^*_{21} & \alpha^*_{22} & \eta^*_2 & \gamma^*_2 \\ 0 & 0 & 0 & 0 & \chi_3 & \chi_4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{103}$$

in the light of (69) to (71). From algebra homomorphism, the coproducts

$$\Delta(\alpha_{ij}) = \sum_{k=1}^{2} \alpha_{ik} \otimes \alpha_{kj} + \sum_{k=1}^{2} \beta_{ik} \otimes \beta^*_{kj}, \tag{104}$$

$$\Delta(\beta_{ij}) = \sum_{k=1}^{2} \alpha_{ik} \otimes \beta_{kj} + \sum_{k=1}^{2} \beta_{ik} \otimes \alpha^*_{kj}, \tag{105}$$

$$\Delta(\eta_i) = \sum_{k=1}^{2} \alpha_{ik} \otimes \eta_k + \sum_{k=1}^{2} \beta_{ik} \otimes \eta^*_k + \eta_i \otimes \chi_3, \tag{106}$$

$$\Delta(\gamma_i) = \sum_{k=1}^{2} \alpha_{ik} \otimes \gamma_k + \sum_{k=1}^{2} \beta_{ik} \otimes \gamma^*_k + \eta_i \otimes \chi_4 + \gamma_i \otimes 1, \tag{107}$$

$$\Delta(\chi_3) = \chi_3 \otimes \chi_3, \tag{108}$$

$$\Delta(\chi_4) = \chi_3 \otimes \chi_4 + \chi_4 \otimes 1, \tag{109}$$

should satisfy the commutation relations (72) to (99). This implies the following additional commutation relation

$$\chi_3\chi_4 = \chi_4\chi_3 \tag{110}$$

which is necessary to complete the algebraic structure of the transformation matrix T.

As in Sect. 2, one can write the transformation matrix T in a block form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} & \eta_1 & \gamma_1 \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} & \eta_2 & \gamma_2 \\ \beta^*_{11} & \beta^*_{12} & \alpha^*_{11} & \alpha^*_{12} & \eta^*_1 & \gamma^*_1 \\ \beta^*_{21} & \beta^*_{22} & \alpha^*_{21} & \alpha^*_{22} & \eta^*_2 & \gamma^*_2 \\ 0 & 0 & 0 & 0 & \chi_3 & \chi_4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{A} & \Upsilon \\ 0 & B \end{pmatrix}, \tag{111}$$

where

$$\mathbb{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \beta^*_{11} & \beta^*_{12} & \alpha^*_{11} & \alpha^*_{12} \\ \beta^*_{21} & \beta^*_{22} & \alpha^*_{21} & \alpha^*_{22} \end{pmatrix}, \tag{112}$$

$$\Upsilon = \begin{pmatrix} \eta_1 & \gamma_1 \\ \eta_2 & \gamma_2 \\ \eta_1^* & \gamma_1^* \\ \eta_2^* & \gamma_2^* \end{pmatrix}, \tag{113}$$

$$B = \begin{pmatrix} \chi_3 & \chi_4 \\ 0 & 1 \end{pmatrix}. \tag{114}$$

Thus, one can write the inverse of the matrix T as

$$T^{-1} = \begin{pmatrix} \mathbb{A}^{-1} & -\mathbb{A}^{-1}\Upsilon B^{-1} \\ 0 & B^{-1} \end{pmatrix}. \tag{115}$$

Since the elements of matrix B commute among themselves, it is easy to write its inverse. But what about the inverse of matrix $\mathbb{A}$ which is the homogeneous part of the transformation matrix T. Since α_{ij}, β_{ij}, α_{ij}^* and β_{ij}^* satisfy the commutation relations written in (72), (73), (78), (90), (91) and (94), one can use Schirrmacher's result [14] related with the multiparametric deformation of $GL(n)$ in order to verify the existence of the inverse of the matrix $\mathbb{A}$. To see the relation between matrix $\mathbb{A}$ and the multiparameter deformed $GL(n)$, let us rewrite the equations written in Schirrmacher's study [14]

$$A_a^i A_b^i = p_{ab} A_b^i A_a^i, \tag{116}$$

$$A_a^i A_a^j = q_{ij} A_a^j A_a^i, \tag{117}$$

$$A_b^i A_a^j = \frac{q_{ij}}{p_{ab}} A_a^j A_b^i, \tag{118}$$

$$A_a^i A_b^j = \frac{p_{ab}}{p_{ij}} A_b^j A_a^i + \left(p_{ab} - \frac{1}{q_{ab}}\right) A_i^b A_j^a, \tag{119}$$

for $i < j$ and $a < b$. Here the upper indices i and j stand for the row number whereas the lower indices a and b stand for the column number of matrix A. These equations coincide with (72), (73), (78), (90), (91) and (94) for

$$q_{12} = p_{12}^{-1} = 1, \tag{120}$$

$$q_{13} = p_{13}^{-1} = q^2, \tag{121}$$

$$q_{14} = p_{14}^{-1} = r^2, \tag{122}$$

$$q_{23} = p_{23}^{-1} = r^2, \tag{123}$$

$$q_{24} = p_{24}^{-1} = q^2, \tag{124}$$

$$q_{34} = p_{34}^{-1} = 1. \tag{125}$$

Thus, we obtain that the homogeneous part of the transformation matrix T is the two parameter deformed general linear group.

If one extends this study from the two boson case to the d boson case, the algebra $\mathcal{A}_d$ can be written as

$$a_i a_i^* - q^2 a_i^* a_i = H, \tag{126}$$

$$a_i H = r^2 H a_i, \tag{127}$$

$$a_i a_j = a_j a_i, \tag{128}$$

$$a_i a_j^* = r^2 a_j^* a_i, \quad i \neq j, \tag{129}$$

where $i, j = 1, 2, \dots, d$. Similar to the two boson case, the symmetry quantum group of this system satisfies (72) to (99) and (110). But in that case all indices can take the values 1 to d, instead of 1 to 2. The homogeneous part of this symmetry group also shows some resemblance to the two boson case such that it becomes a two parameter deformed general linear group obtained from $GL_{q_{ij},p_{ij}}(2d)$ where

$$q_{ij} = p_{ij}^{-1} = \begin{cases} 1 & i < j \leq d \text{ or } d < i < j \leq 2d \\ q^2 & i < d \text{ and } j = i + d \\ q^2 & i = d \text{ and } j = 2d \\ r^2 & i < d \text{ and } d < j \leq 2d \text{ and } j \neq i + d. \\ r^2 & i = d \text{ and } d < j < 2d. \end{cases}$$

4 Conclusion

In this work, we obtained the inhomogeneous quantum symmetry group of the two parameter deformed boson algebra defined by (126) to (129). Since the homogeneous part of this group is $GL_{q_{ij},p_{ij}}(2d)$ with special p_{ij} and q_{ij} values, it is possible to call the symmetry group of this system as the Bosonic Inhomogeneous multiparameter deformed General Linear Quantum Group, $BIGL_{q,r}(2d)$. It is obvious that this quantum group includes only two independent deformation parameters which are q^2 and r^2. If one considers the case where q is equal to r, the quantum group reduces the inhomogeneous quantum invariance group of the multi-dimensional q-deformed Bosonic Newton oscillator algebra [26].

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